MATH 3060 Assignment 8 solution

Chan Ki Fung

December 15, 2021

1. This follows from the following two facts:

   - The set of algebraic numbers is countable.
   - The complement of a countable subset of $\mathbb{R}$ is dense.

   For the first item, note that for each fix integer $d > 0$, the set S_d of integral polynomials with degree $\leq d$ is countable. And the set of all integral polynomials is $\cup_d S_d$, which is also countable. For each integral polynomial $f \not\equiv 0$, the set of roots Z_f is finite. Now, the set of algebraic numbers is equal to the set

   $$\cup_{f \text{ nonzero, integral}} Z_f,$$

   which is countable.
   For the second item, let $Q \subset \mathbb{R}$ be countable, then

   $$\mathbb{R} \setminus Q = \cap_{q \in Q} (\mathbb{R} \setminus \{q\})$$

   is a countable intersection of open dense subsets of $\mathbb{R}$, and is itself dense by Baire' category theorem.

2. Let $||\cdot||$ be an arbitrary norm of $\mathbb{R}^n$.Denote e_i the standard basis of $\mathbb{R}^n$. Let $M = \max_i\{||e_i||\}$, then for any $x = \sum_i x_i e_i \in \mathbb{R}^n$, we have

   $$||x|| \leq M \sum_i |x_i| \leq M \sqrt{\sum_i x_i^2} \sqrt{\sum_i 1^2} = \sqrt{n} M ||x||_2.$$

   This proves $||\cdot||$ is weaker than $||\cdot||_2$, we need to prove the converse. Suppose $||\cdot||$ is not stronger than $||\cdot||_2$, then we can find $0 \neq x_n \in \mathbb{R}$ so that $||x_n||_2 > n||x_n||$. Replacing x_n with $\frac{x_n}{||x_n||_2}$, we may assume $||x_n||_2 = 1$, and thus assume $x_n \to x$ with respect to $||\cdot||_2$, in particular $||x||_2 = 1$ and $x \neq 0$.
   Consider the identity map $\mathrm{id} : (\mathbb{R}^n, ||\cdot||_2) \to (\mathbb{R}^n, ||\cdot||)$, we know this is a continuous map because $||\cdot||$ is weaker than $||\cdot||_2$, so we must have, $x_n \to x$ with respect to $||\cdot||$, but then $0 \neq ||x|| = \lim ||x_n|| \leq \lim \frac{1}{n} = 0$, which is absurd.

3. Since $G_i \supset \cap G_i$, G_i is open dense, and G_i^c is nowhere dense. Therefore,

$$G^c = \cup G_i^c$$

is a union of nowhere dense subsets of $\mathbb{R}$, and is thus of first category. So G is residual.

4. First we show $C_{\mathbb{Q}}$ is dense. In fact, let (x_n) be a sequence converges to x and $\epsilon > 0$, we can find $x' \in \mathbb{Q}$ with $|x' - x| < \epsilon$. If we define (x'_n) by $x'_n = x_n + x' - x$, then x'_n converges to $x' \in \mathbb{Q}$ and $d_\infty((x_n), (x'_n)) < \epsilon$. This shows that $C_{\mathbb{Q}}$ is dense, and in particular it is not nowhere dense.

   Next we show that $C_{\mathbb{Q}}$ is of first category. In fact, we can write

$$C_{\mathbb{Q}} = \cup_{q \in \mathbb{Q}} C_q$$

   where

$$C_q = \{(x_n) \in C : \lim x_n = q\}.$$

   It suffices to show each C_q is nowhere dense. Note that C_q is closed because if $(x_n) \notin C_q$, then $\epsilon = |\lim x_n - q| > 0$, and then $(x'_n) \notin C_q$ for any (x'_n) with $d_\infty((x_n), (x'_n)) < \epsilon$.
   Next we show that C_q has no interior points. In fact, let $(x_n) \in C_q$, then $\lim x_n = q$. For any $\epsilon > 0$, if we define (x'_n) by $x'_n = x_n + \epsilon/2$, then $\lim x'_n \neq q$, and so $x' \notin C_q$.